Coordinate Transformation and Truncation for Rotating Spacecraft with Flexible Appendages

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The analytical structures of alternative coordinate transformations for the variables that characterize the deformations of finite element models of flexible appendages on rotating spacecraft are examined here. Particular emphasis is given to the truncation operations that are essential for efficient numerical simulations of flexible spacecraft, due to the necessarily large number of coordinates employed in a finite element description. Certain of the coordinate transformations in the literature require matrix inversions to obtain explicit transformed equations of the desired uncoupled form, and these inversions significantly complicate the truncation process. While mathematical rigor demands that the inversions be performed and the explicit uncoupled form obtained before truncation, computational limitations require that, in practice, truncation of the transformation matrix be accomplished prior to inversion, necessitating, in these cases, the adoption of a pseudo-inverse. The central new contribution of this paper is a theorem which establishes necessary and sufficient conditions for the commutativity of the required truncation and inversion operations. The application of this theorem jeopardizes the utility of two of the four previously published transformation procedures considered here. The recommended alternative for the rotating elastic appendage is a complex transformation with previously published orthogonality properties, which permit the derivation in this paper of a relatively simple explicit set of transformed and truncated equations of modal vibration in real variables.

Introduction

It is now commonplace in dynamic simulation practice to characterize spacecraft with flexible appendages in terms of hybrid coordinates, consisting of discrete coordinates for rigid bodies in the system and distributed or modal coordinates for the flexible appendages. The selection of mode shapes and frequencies associated with the modal coordinates is usually accomplished by adopting a finite element model of the appendage and writing equations of motion for the small vibrations of the appendage with respect to a base which has a nominally constant (perhaps zero) inertial angular velocity; the eigenvectors of these equations then provide the mode shapes and the eigenvalues provide the natural frequencies of appendage vibration. This information is used in various ways to construct a transformation matrix which transforms the hundreds, or even thousands, of discrete nodal coordinates of the finite element model into distributed coordinates; these then are reduced in number by a process of coordinate truncation, which sacrifices mathematical rigor for computational feasibility (hopefully without doing violence to the salient features of the mathematical model). The transformed and truncated equations of appendage vibrations are then combined with translational and rotational equations of the total vehicle and equations which characterize the control system, to obtain a set of ordinary differential equations for numerical integration in spacecraft simulation. This paper addresses an aspect of the truncation problem which arises in the case of modal analysis of an elastic appendage on a rotating base.

We will critically examine four previously published alternative transformation procedures, three of which involve the inversion of a matrix. Whereas mathematical arguments demand that this matrix be inverted prior to truncation, practical considerations demand that truncation precede inversion (which then may require a pseudo-inverse). We will establish necessary and sufficient conditions for the commutativity of matrix truncation and inversion to define the formal limits of these procedures. We will conclude with a recommendation for the selection of a previously published coordinate transformation for elastic appendages on rotating bodies, and a detailed representation of the real, first-order equations which emerge from the recommended complex transformation.

As already shown,[1] the equations of small vibration of a finite element model of a flexible appendage on a rigid rotating base have the structure

$$M\ddot{q}+D\dot{q}+G\dot{q}+Kq+Aq=F \tag{1}$$

where for an appendage with n nodes q is the $6n\times 1$ matrix of small translational and rotational deformations relative to a nominal (deformed) state, the matrices M, D, and K are symmetric, and the matrices G and A are skew-symmetric, with M positive definite and hence nonsingular. These equations are applicable whether the appendage mass is concentrated into rigid nodal bodies[2] or also distributed over the finite elements of the model.[1] The matrix F in Eq. (1) holds the vibration forcing functions, which include deviations of the base motion from a state of steady spin, but the coefficient matrices in Eq. (1) also depend on the inertial angular velocity of the base. If, and only if, this velocity experiences onl small deviations from a nominal constant can these coefficient matrices be treated formally (after linearization) as constants, and in this case the matrix A also disappears for realistic idealizations of the appendage.[3] In most engineering applications, the damping matrix D is also ignored for the purposes of coordinate transformation (modal analysis) under investigation in this paper. This seems to be an acceptable engineering practice as long as all energy dissipation is attributable to distributed structural damping rather than discrete energy absorbers, such as fluid dashpots in the system.

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Of course, Eq. (1) is of a form which has much wider applicability than that cited here. As may be established from any text on analytical dynamics (see for example, Whittaker[4]) this is the form of the equations of small vibration of any discretized mechanical system, and, in particular, this is the form of the linearized equations of the *total* spacecraft, not only of a flexible appendage on a rotating body. The emphasis here on a particular application is a reflection of the practical motivation of this study, but the mathematical results are applicable to any process characterized by Eq. (1).

In practical applications the dimension of q often numbers in the hundreds, or even thousands, and this generally precludes the direct incorporation of this equation into a simulation of a rotating system, such as a spacecraft. Instead one must transform the nodal displacement variables q into distributed or modal coordinates and then somehow justify the deletion of most of the variables by coordinate truncation. The usual procedure is to seek a transformation that, in some sense, uncouples the new vibration coordinates from each other in the homogeneous counterpart to Eq. (1). The independence of these coordinates permits their individual consideration as candidates for truncation or retention in the reduced-dimension equations of appendage vibration. In the next two sections, such transformations are presented and evaluated in the light of a commutativity theorem to be developed.

Transformations for Damped Rotating Appendages

For the most general problem of small-deformation flexible appendages on a base rotating at constant nominal speed, the coefficient matrices in Eq. (1) are all constant but nonzero, and for the desired coordinate transformation one must rewrite Eq. (1) in terms of the state variables in the $12n \times 1$ matrix

$$Q \triangleq \begin{bmatrix} q \\ \hline \dot{q} \end{bmatrix} \tag{2}$$

as

$$\dot{Q} = BQ + L \tag{3}$$

where

$$B \triangleq \left[\begin{array}{c|c} 0 & U_{6n} \\ \hline -M^{-1}(K+A) & -M^{-1}(G+D) \end{array}\right] \tag{4}$$

and

$$L \triangleq \begin{bmatrix} 0 \\ \hline M^{-1}F \end{bmatrix} \tag{5}$$

As is well known, the homogeneous counterpart to Eq. (3) has the general solution

$$Q = \Phi e^{\Lambda t} Q_0 \tag{6}$$

for some $12n \times 1$ constant matrix Q_0, where the columns $\Phi^1,..,\Phi^{12n}$ of Φ are the system eigenvectors, assumed here to be linearly independent,‡ and Λ is the diagonal matrix of corresponding eigenvalues $\lambda_1,\ldots,\lambda_{12n}$, so that $e^{\Lambda t}$ is the diagonal matrix of scalars $e^{\lambda_1 t},\ldots,e^{\lambda_{12n} t}$. Thus

$$|B - \lambda_j U_{12n}| = 0 \quad j = 1,\ldots,12n \tag{7}$$

and

$$\Lambda = \Phi^{-1} B \Phi \tag{8}$$

‡The validity of this assumption is assured if the eigenvalues are distinct[5]; if the eigenvectors are dependent, the solution contains powers of t, implying at least for the (undamped) case with $D = A = 0$ an unacceptable instability (see Ref. 2, p. 43).

If now Φ' and Λ' similarly contain the eigenvectors and eigenvalues of the adjoint equation

$$\dot{Q}' = B^T Q' \tag{9}$$

then

$$|B^T - \lambda_j' U_{12n}| = 0 \quad j = 1,\ldots,12n$$

so that

$$|B - \lambda_j' U_{12n}| = 0 \quad j = 1,\ldots,12n$$

and thus

$$\Lambda' = \Lambda \tag{10}$$

In parallel with Eq. (8) we have, after noting the diagonality of Λ' and the commutativity of matrix inversion and transposition

$$\Lambda' = (\Phi')^{-1} B^T \Phi' = \Lambda'^T = \Phi'^T B (\Phi'^T)^{-1}$$

In conjunction with Eqs. (10) and (8), this implies

$$\Phi^{-1} B \Phi = \Phi'^T B (\Phi'^T)^{-1}$$

so that, in conformity with the more general result in Wilkinson,[6]

$$\Phi^{-1} = \ell \Phi'^T \tag{11}$$

for some $12n \times 12n$ diagonal matrix ℓ which depends upon the normalization of Φ and Φ'.

Substitution of the transformation

$$Q = \Phi Y \tag{12}$$

into Eq. (3) and premultiplication by Φ^{-1} produces, in view of Eq. (8), the desired homogeneously uncoupled equations

$$\dot{Y} = \Lambda Y + \Phi^{-1} L \tag{13}$$

The calculation of Φ^{-1} in Eq. (13) can be avoided with the substitution of Eq. (11), with the result

$$\dot{Y} = \Lambda Y + \ell \Phi'^T L \tag{14}$$

but this option requires the calculation of the adjoint system eigenvectors in Φ'.

As noted previously, practical considerations mandate the truncation of the transformed coordinates, so that the $12n \times 1$ matrix Y must, in some sense, be replaced by the truncated matrix $\bar{Y}$, with dimensions identified here as $2N \times 1$. (Typically N is of dimension between 1 and 20.) If the matrix F in Eq. (1) were wholly independent§ of q, then no mathematical compromise would be involved in the isolation for further consideration of $2N$ of the $12n$ scalar equations represented by Eqs. (13) and (14), since each set of $12n$ equations would then be uncoupled. We could accomplish this isolation by selecting $2N$ rows of either of the matrix Eqs. (13) or (14); if we use an overbar to distinguish a truncated matrix we can record the truncated equations as

$$\dot{\bar{Y}} = \bar{\Lambda}\, \bar{Y} + (\overline{\Phi^{-1}}) L \tag{15}$$

and

$$\dot{\bar{Y}} = \bar{\Lambda}\, \bar{Y} + \bar{\ell} \bar{\Phi}'^T L \tag{16}$$

§In hybrid coordinate dynamic analysis, this hypothesis may be compromised by the fact that Eq. (1) provides only a subset of the system equations.

In Eq. (15) the matrix $(\overline{\Phi^{-1}})$ consists of the first $2N$ rows of Φ^{-1}, but for systems of typical dimension it is not feasible to calculate all of the hundreds or even thousands of eigenvectors which comprise Φ, and neither Φ nor Φ^{-1} is available. To circumvent this problem, it was proposed earlier[2] that Eq. (15) be replaced by

$$\dot{\bar{Y}} = \bar{\Lambda}\ \bar{Y} + \bar{\Phi}^{\dagger} L \tag{17}$$

where

$$\bar{\Phi}^{\dagger} \triangleq (\bar{\Phi}^T \bar{\Phi})^{-1} \bar{\Phi}^T$$

is a pseudo-inverse of the $12n \times 2N$ matrix $\bar{\Phi}$, which can be obtained by calculating only the $2N$ eigenvectors comprising the columns of $\bar{\Phi}$, without even computing all of the elements of Φ. Equations (17) and (15) are equivalent only if the inversion and truncation of Φ are commutative operations.

Conditions for Commutativity of Truncation and Inversion

Our central purpose is to develop and apply the commutativity criteria required to test the equivalance of alternative forms of truncated, transformed equations of appendage vibration, as illustrated previously by the comparison of Eqs. (15) and (17), and as required also for the evaluation of other transformations to be considered subsequently.

In what follows, we will first establish a theorem providing the required commutativity conditions, and then use this result to test alternative transformation procedures applicable, firstly, to the general problem of damped, rotating, finite element models, and, secondly, to undamped (elastic), rotating, finite element models. Explicit results in the form of real equations are provided in the latter case.

Theorem: For any nonsingular square matrix m partitioned as

$$m = \left[\begin{array}{c:c} a & b \\ \hdashline c & d \end{array}\right] \tag{18}$$

where a and d are square, the condition

$$a^T b + c^T d = 0 \tag{19}$$

is necessary and sufficient for the equivalence

$$\bar{m}^{\dagger} = (\overline{m^{-1}}) \tag{20}$$

where $\bar{m}$ is the column partition of m defined by

$$\bar{m} \triangleq \left[\begin{array}{c} a \\ \hdashline c \end{array}\right] \tag{21}$$

and where $(\overline{m^{-1}})$ is the uppermost row-partition of m^{-1} having the dimensions of $\bar{m}^T$. Here $\bar{m}^{\dagger}$ designates the pseudo-inverse

$$\bar{m}^{\dagger} = (\bar{m}^T \bar{m})^{-1}\ \bar{m}^T \tag{22}$$

Proof: See Appendix A.

Corollary: For any nonsingular square matrix m partitioned as

$$m = \left[\begin{array}{c:c} a_l & b_l \\ \hdashline c_l & d_l \end{array}\right] \tag{23}$$

where a_l has dimension $p \times r$ for $p \geq r$, the condition

$$a_l^T b_l + c_l^T d_l = 0 \tag{24}$$

is necessary and sufficient for the equivalence

$$\bar{m}^{\dagger} = (\overline{m^{-1}}) \tag{25}$$

where $\bar{m}$ is the column partition of m defined by

$$\bar{m} \triangleq \left[\begin{array}{c} a_l \\ \hdashline c_l \end{array}\right] \tag{26}$$

and where $\bar{m}^{\dagger}$ and $(\overline{m^{-1}})$ are as defined following Eq. (21).

Proof: See Appendix B.

Application of Commutativity Theorem to Φ

Comparison of Eqs. (2) and (6) indicates that the columns of Φ must have the structure

$$\Phi^{\alpha} = \left[\begin{array}{c} \phi^{\alpha} \\ \hdashline \phi^{\alpha}\lambda_{\alpha} \end{array}\right] \tag{27}$$

If λ_{α} is a solution of Eq. (7), so also must be its complex conjugate λ_{α}^* (since B is real), and corresponding eigenvectors must also appear in conjugate pairs. With the (physically imperative) assumption that conjugate modes are *together* either truncated or retained, one can partition Φ as follows

$$\phi = \left[\begin{array}{c:c:c:c} \bar{\phi} & \bar{\phi}^* & \bar{\bar{\phi}} & \bar{\bar{\phi}}^* \\ \hdashline \bar{\phi}\bar{\lambda} & \bar{\phi}^*\bar{\lambda}^* & \bar{\bar{\phi}}\bar{\bar{\lambda}} & \bar{\bar{\phi}}^*\bar{\bar{\lambda}}^* \end{array}\right] \tag{28}$$

where, in comparison with Eq. (23)

$$a_l = [\bar{\phi} \ \vdots\ \bar{\phi}^*]$$

$$b_l = [\bar{\bar{\phi}} \ \vdots\ \bar{\bar{\phi}}^*]$$

$$c_l = [\bar{\phi}\bar{\lambda} \ \vdots\ \bar{\phi}^*\bar{\lambda}^*]$$

$$d_l = [\bar{\bar{\phi}}\bar{\bar{\lambda}} \ \vdots\ \bar{\bar{\phi}}^*\bar{\bar{\lambda}}^*]$$

Imposition of the commutativity condition corollary in Eq. (24) requires that

$$\left[\begin{array}{c:c} \bar{\phi}^T\bar{\bar{\phi}} + \bar{\lambda}\bar{\phi}^T\bar{\bar{\phi}}\bar{\bar{\lambda}} & \bar{\phi}^T\bar{\bar{\phi}}^* + \bar{\lambda}\bar{\phi}^T\bar{\bar{\phi}}^*\bar{\bar{\lambda}}^* \\ \hdashline \bar{\phi}^{*T}\bar{\bar{\phi}} + \bar{\lambda}^*\bar{\phi}^{*T}\bar{\bar{\phi}}\bar{\bar{\lambda}} & \bar{\phi}^{*T}\bar{\bar{\phi}}^* + \bar{\lambda}^*\bar{\phi}^{*T}\bar{\bar{\phi}}^*\bar{\bar{\lambda}}^* \end{array}\right] = 0$$

which is equivalent to the combination

$$\bar{\phi}^T\bar{\bar{\phi}} + \bar{\lambda}\bar{\phi}^T\bar{\bar{\phi}}\bar{\bar{\lambda}} = 0$$

and

$$\bar{\phi}^T\bar{\bar{\phi}}^* + \bar{\lambda}\bar{\phi}^T\bar{\bar{\phi}}^*\bar{\bar{\lambda}}^* = 0$$

or in scalar terms (with no summation convention implied)

$$\bar{\phi}^{\alpha T}\bar{\bar{\phi}}^{\beta}(1 + \bar{\lambda}_{\alpha}\bar{\bar{\lambda}}_{\beta}) = 0 \tag{29}$$

and

$$\bar{\phi}^{\alpha T}\bar{\bar{\phi}}^{\beta *}(1 + \bar{\lambda}_{\alpha}\bar{\bar{\lambda}}_{\beta}^*) = 0 \tag{30}$$

The quantities in parentheses in Eqs. (29) and (30) cannot both be zero unless both $\bar{\lambda}_{\alpha}$ and $\bar{\bar{\lambda}}_{\beta}$ are real and their product is minus one. If this intolerable constraint is rejected, the commutativity condition for this truncation becomes

$$\bar{\phi}^T\bar{\bar{\phi}} = 0 \quad \text{and} \quad \bar{\phi}^T\bar{\bar{\phi}}^* = 0 \tag{31}$$

If the eigenvectors are written in terms of real and imaginary parts, with the symbol definitions

$$\bar{\phi} \triangleq \bar{\psi} + i\bar{\gamma} \quad \text{and} \quad \bar{\bar{\phi}} \triangleq \bar{\bar{\psi}} + i\bar{\bar{\gamma}} \tag{32}$$

then Eqs. (31) are equivalent to the real equations

$$\bar{\psi}^T \bar{\bar{\psi}} = 0 \tag{33a}$$

$$\bar{\gamma}^T \bar{\bar{\gamma}} = 0 \tag{33b}$$

$$\bar{\psi}^T \bar{\bar{\gamma}} = 0 \tag{33c}$$

$$\bar{\gamma}^T \bar{\bar{\psi}} = 0 \tag{33d}$$

The orthogonality conditions in Eq. (32) then become necessary and sufficient for the commutativity of truncation and inversion. These conditions are quite stringent, but they can be circumvented in special cases, as shown in the following argument.

The partitioning of Φ indicated in Eq. (28) is not obligatory, nor is the choice of the definition of Q in Eq. (2). For an appendage with appropriate symmetry properties it may be advantageous to replace Eq. (2) by

$$\tilde{Q} \triangleq [q_1 \dot{q}_1 q_2 \dot{q}_2 \ldots q_{6n} \dot{q}_{6n}]^T \tag{34}$$

in which $q_1,\ldots,q_{3n}$ and $q_{3n+1},\ldots,q_{6n}$ are coordinates defining parallel motions of appendage nodes located symmetrically with respect to a central rigid body. For this special case the appendage mode shapes will be either symmetric with respect to the central body or antisymmetric. If one wishes to truncate the symmetric modes (as would be appropriate for a spacecraft attitude control simulation), then instead of the partitioning in Eq. (28), we could write the transformation matrix $\tilde{\Phi}$ of eigenvectors as

$$\tilde{\Phi} = \left[\begin{array}{c|c} A & S \\ \hline -A & S \end{array}\right] \tag{35}$$

where the left column partitions (comprising $\bar{\bar{\Phi}}$) contain the antisymmetric modes and the right column partitions contain the symmetric modes. Now after identification of Eqs. (35) and (18) we find that our commutativity condition in Eq. (19) becomes

$$A^T S - A^T S = 0$$

so that reversal of the sequence of inversion and truncation operations on $\tilde{\Phi}$ is correct, and the equivalents of Eq. (15) and (16) are identical.

For the more general case, however, Eqs. (15) and (16) are not identical, since Eqs. (33) are not satisfied. Since we cannot, in practice, hope to obtain Φ as required by Eq. (15), and we have no sound argument for the adoption of Eq. (17), we must rely upon Eq. (16), despite the implied necessity of computing $2N$ of the adjoint system eigenvectors.

In the next section we will consider the important special case of the rotating elastic appendage with no explicit damping, to seek out simplifications stemming from the absence of D and A in the appendage vibration Eq. (1).

Real Transformation for Rotating Elastic Appendages

When Eq. (1) is replaced by

$$M\ddot{q} + G\dot{q} + Kq = F \tag{36}$$

the system eigenvalues for a stable system become imaginary and may be designated in conjugate pairs as $\pm i\sigma_\alpha$ for $\alpha = 1,\ldots,6n$. The corresponding eigenvectors are, however, generally complex, suggesting the designations

$$\Phi^\alpha = \Psi^\alpha + i\Gamma^\alpha \quad \text{and} \quad \phi^\alpha = \psi^\alpha + i\gamma^\alpha \tag{37}$$

for the matrices Φ^α and ϕ^α appearing in Eq. (27). In these terms, Eq. (27) implies

$$\Psi^\alpha = \left[\begin{array}{c} \psi^\alpha \\ \hline -\gamma^\alpha \sigma_\alpha \end{array}\right] \quad \text{and} \quad \Gamma^\alpha = \left[\begin{array}{c} \gamma^\alpha \\ \hline \psi^\alpha \sigma_\alpha \end{array}\right] \tag{38}$$

The eigenproblem associated with Eq. (36) is treated in the classical literature (see Whittaker,[4] pp. 427-429) and computational aspects of the problem are receiving much attention in the current literature (see, for example Refs. 7-10). In this paper we examine alternative transformations from the literature to assess them for the desired property of truncation-inversion commutativity,

If, in Eq. (36), we have $F = 0$, we can without contradiction substitute

$$q = \psi z + \gamma\sigma^{-1}\dot{z} \tag{39a}$$

$$\dot{q} = -\gamma\sigma z + \psi\dot{z} \tag{39b}$$

$$\ddot{q} = -\gamma\sigma\dot{z} + \psi\ddot{z} \tag{39c}$$

to obtain

$$M(-\gamma\sigma\dot{z} + \psi\ddot{z}) + G(-\gamma\sigma z + \psi\dot{z}) + K(\psi z + \gamma\sigma^{-1}\dot{z}) = 0$$

or

$$M\psi\ddot{z} + (-M\gamma\sigma + G\psi + K\gamma\sigma^{-1})\dot{z} + (-G\gamma\sigma + K\psi)z = 0 \tag{40}$$

A homogeneous solution of Eq. (36) is

$$q_h = (\psi^\alpha + i\gamma^\alpha)e^{i\sigma_\alpha t} \tag{41}$$

which when substituted yields

$$M(\psi^\alpha + i\gamma^\alpha)(-\sigma_\alpha^2) + G(\psi^\alpha + i\gamma^\alpha)(i\sigma_\alpha) + K(\psi^\alpha + i\gamma^\alpha) = 0$$

or

$$-M\psi^\alpha\sigma_\alpha^2 - G\gamma^\alpha\sigma_\alpha + K\psi^\alpha = 0 \tag{42a}$$

and

$$-M\gamma^\alpha\sigma_\alpha^2 + G\psi^\alpha\sigma_\alpha + K\gamma^\alpha = 0 \tag{42b}$$

and Eqs. (42a) and (42b) considered for $\alpha = 1,\ldots,6n$ imply

$$-M\psi\sigma^2 - G\gamma\sigma + K\psi = 0 \tag{43a}$$

$$-M\gamma\sigma + G\psi + K\gamma\sigma^{-1} = 0 \tag{43b}$$

Thus, with Eq. (43b), Eq. (40) reduces to

$$M\psi\ddot{z} + (-G\gamma\sigma + K\psi)z = 0$$

which, after premultiplication by ψ^T, becomes

$$\psi^T M\psi\ddot{z} + (\psi^T K\psi - \psi^T G\gamma\sigma)z = 0$$

and with ψ^T times Eq. (43a) this is

$$\psi^T M\psi(\ddot{z} + \sigma^2 z) = 0$$

or with nonsingular $\psi^T M\psi$

$$\ddot{z} + \sigma^2 z = 0 \tag{44}$$

This same result arises if Eq. (39b) is equated to the derivative of Eq. (39a), so that contradiction is avoided. The transformations in Eq. (39) are *not* acceptable in the inhomogeneous case of Eq. (36) of primary interest here, however, because these transformations directly imply Eq. (44). Instead Eq. (36) must in the inhomogeneous case be written in first-order form, perhaps as in Eq. (3). In this case one can accomplish the desired transformation to uncoupled (pairs of) scalar equations with the *real* transformation

$$Q=PZ \tag{45}$$

where

$$P \triangleq \left[\begin{array}{c|c} \psi & \gamma \\ \hline -\gamma\sigma & \psi\sigma \end{array}\right] \tag{46}$$

and P^{-1} exists if the eigenvectors of the system are independent, as required for stability in this undamped case. Substituting Eq. (45) into Eq. (3) with $D=A=0$ and premultiplying by P^{-1} produces

$$\dot{Z} = \left[\begin{array}{c|c} 0 & \sigma \\ \hline -\sigma & 0 \end{array}\right] Z + P^{-1}L \tag{47}$$

as may be confirmed by considering the homogeneous case and defining for this case

$$Z = \left[\begin{array}{c} z \\ \hline \sigma^{-1}\dot{z} \end{array}\right]$$

Equation (47) offers an advantage over Eq. (13) and (14) in that the latter involve complex numbers in Y, Λ, Φ, and Φ'. The inversion of P in Eq. (47) becomes particularly critical when (as required for practical structural dynamics) this inversion is either to be followed by, or to precede, coordinate truncation. We must once again face the critical question raised in this paper, "Are truncation and inversion commutative?" More explicitly, we must determine whether or not the truncation of Eq. (47) to

$$\dot{\bar{Z}} = \left[\begin{array}{c|c} 0 & \bar{\sigma} \\ \hline -\bar{\sigma} & 0 \end{array}\right] \bar{Z} + \overline{(P^{-1})}L \tag{48}$$

provides the same result as would be obtained by substituting

$$Q = \bar{P}\bar{Z} \triangleq \left[\begin{array}{c|c} \bar{\psi} & \bar{\gamma} \\ \hline -\bar{\gamma}\bar{\sigma} & \bar{\psi}\bar{\sigma} \end{array}\right] \bar{Z} \tag{49}$$

into Eq. (3) with $D=A=0$ and premultiplying by

$$\bar{P}^{\dagger} \triangleq (\bar{P}^T\bar{P})^{-1}\bar{P}^T$$

to obtain

$$\dot{\bar{Z}} = \left[\begin{array}{c|c} 0 & \bar{\sigma} \\ \hline -\bar{\sigma} & 0 \end{array}\right] \bar{Z} + \bar{P}^{\dagger}L \tag{50}$$

This issue was not confronted in Ref. 2, where an affirmative answer was presumed. The presumed equivalence of Eqs. (49) and (50) requires for its validity that

$$\bar{P}^{\dagger}B\bar{P} = \left[\begin{array}{c|c} 0 & \bar{\sigma} \\ \hline -\bar{\sigma} & 0 \end{array}\right] \tag{51}$$

and

$$\bar{P}^{\dagger} = \overline{(P^{-1})} \tag{52}$$

Equation (51) is proved in an appendix of the referenced report that underlies Ref. 1, but the validity of Eq. (52) had not been examined previously.

To apply the commutativity conditions in Eq. (24) to this transformation we must rearrange the columns of P shown in Eq. (46) so as to collect those to be preserved in the left partition. If Eq. (46) is written as

$$P \triangleq \left[\begin{array}{c|c|c|c} \bar{\psi} & \bar{\bar{\psi}} & \bar{\gamma} & \bar{\bar{\gamma}} \\ \hline -\bar{\gamma}\bar{\sigma} & -\bar{\bar{\gamma}}\bar{\bar{\sigma}} & \bar{\psi}\bar{\sigma} & \bar{\bar{\psi}}\bar{\bar{\sigma}} \end{array}\right] \tag{53}$$

then the desired truncated transformation matrix is

$$\bar{P} \triangleq \left[\begin{array}{c|c} \bar{\psi} & \bar{\gamma} \\ \hline -\bar{\gamma}\bar{\sigma} & \bar{\psi}\bar{\sigma} \end{array}\right] \tag{54}$$

so that to apply our commutativity theorem we must start with the transformation matrix

$$\tilde{P} \triangleq \left[\begin{array}{c|c|c|c} \bar{\psi} & \bar{\gamma} & \bar{\bar{\psi}} & \bar{\bar{\gamma}} \\ \hline -\bar{\gamma}\bar{\sigma} & \bar{\psi}\bar{\sigma} & -\bar{\bar{\gamma}}\bar{\bar{\sigma}} & \bar{\bar{\psi}}\bar{\bar{\sigma}} \end{array}\right] \tag{55}$$

rather than with P. In terms of the interchange operator V defined by Eq. (A5) in Appendix A, we have

$$P = \tilde{P}V \tag{56}$$

so that $\tilde{P}$ differs from P only in the sequence of the transformed scalar equations it produces.

Now the commutativity test in Eq. (24) requires that

$$[\bar{\psi} \mid \bar{\gamma}]^T[\bar{\bar{\psi}} \mid \bar{\bar{\gamma}}] + [-\bar{\gamma}\bar{\sigma} \mid \bar{\psi}\bar{\sigma}]^T[-\bar{\bar{\gamma}}\bar{\bar{\sigma}} \mid \bar{\bar{\psi}}\bar{\bar{\sigma}}] = 0$$

or

$$\bar{\psi}^T\bar{\bar{\psi}} + \bar{\sigma}\bar{\gamma}^T\bar{\bar{\gamma}}\bar{\bar{\sigma}} = 0 \tag{57a}$$

$$\bar{\psi}^T\bar{\bar{\gamma}} - \bar{\sigma}\bar{\gamma}^T\bar{\bar{\psi}}\bar{\bar{\sigma}} = 0 \tag{57b}$$

$$\bar{\gamma}^T\bar{\bar{\psi}} - \bar{\sigma}\bar{\psi}^T\bar{\bar{\gamma}}\bar{\bar{\sigma}} = 0 \tag{57c}$$

$$\bar{\gamma}^T\bar{\bar{\gamma}} + \bar{\sigma}\bar{\psi}^T\bar{\bar{\psi}}\bar{\bar{\sigma}} = 0 \tag{57d}$$

Equations (57a) and (57d) combine to provide

$$\bar{\psi}^T\bar{\bar{\psi}} - \bar{\sigma}^2\bar{\psi}^T\bar{\bar{\psi}}\bar{\bar{\sigma}}^2 = 0$$

and Eqs. (57b) and (57c) provide

$$\bar{\psi}^T\bar{\bar{\gamma}} - \bar{\sigma}^2\bar{\psi}^T\bar{\bar{\gamma}}\bar{\bar{\sigma}}^2 = 0$$

Satisfaction of Eqs. (57) thus requires either the orthogonality conditions in Eqs. (33), or the unacceptable constraint

$$\bar{\sigma}_{\alpha}^2\bar{\bar{\sigma}}_{\beta}^2 = 1 \tag{58}$$

Thus, the computationally attractive option of using Eq. (50) is not generally equivalent to the computationally infeasible option of using Eq. (48), although the latter bears a more formal mathematical relationship to the exact Eq. (47).

Complex Transformation for Rotating Elastic Appendages

In what follows we examine a different transformation of the rotating, elastic appendage vibration equations [Eq. (36)], to circumvent the inversion/truncation problems that jeopardize the options represented by Eqs. (50) and (17) as practical truncated versions of the transformed Eqs. (48) and (13), respectively.

As an alternative to the first-order form in Eq. (3), one can write Eq. (36) in the first-order form

$$\mathcal{A}\dot{Q} + \mathcal{B}Q = \mathcal{L} \tag{59}$$

where

$$\mathcal{A} \triangleq \left[\begin{array}{c|c} K & 0 \\ \hline 0 & M \end{array}\right] \qquad \mathcal{B} \triangleq \left[\begin{array}{c|c} 0 & -K \\ \hline K & G \end{array}\right] \qquad \mathcal{L} \triangleq \left[\begin{array}{c} 0 \\ \hline F \end{array}\right]$$

so that

$$\mathcal{A}^T = \mathcal{A} \text{ and } \mathcal{B}^T = -\mathcal{B} \tag{60}$$

Equations (59) and (3) are equivalent equations written in terms of the same state variable Q; they have the same eigenvalues λ_α and eigenvectors Φ^α, $\alpha=1,\ldots,12n$, so that

$$[\mathcal{A}\lambda_\alpha+\mathcal{B}]\Phi^\alpha=0 \quad \alpha=1,\ldots,12n \tag{61}$$

and, with the application to Eq. (59) of the same transformation $Q=\Phi Y$ applied previously to Eq. (3), we find

$$\mathcal{A}\triangleq\left[\begin{array}{c|c} K & 0 \\ \hline 0 & M \end{array}\right] \qquad \mathcal{B}\triangleq\left[\begin{array}{c|c} 0 & -K \\ \hline K & G \end{array}\right] \qquad \mathcal{L}\triangleq\left[\begin{array}{c} 0 \\ \hline F \end{array}\right]$$

$$\mathcal{A}\Phi\dot{Y}=-\mathcal{B}\Phi Y+\mathcal{L}$$

This result can be premultiplied by the transposed conjugate of Φ, and the result premultiplied by $(\Phi^{*T}\mathcal{A}\Phi)^{-1}$, to obtain

$$\dot{Y}=-(\Phi^{*T}\mathcal{A}\Phi)^{-1}(\Phi^{*T}\mathcal{B}\Phi)Y+(\Phi^{*T}\mathcal{A}\Phi)^{-1}\mathcal{L} \tag{62}$$

Meirovitch[10] has established the orthogonality condition

$$\Phi^{\beta*T}\mathcal{A}\,\Phi^\alpha=0 \quad \text{for} \quad \lambda_\alpha\neq\lambda_\beta \tag{63}$$

so that for a system with distinct eigenvalues the matrix $\Phi^{*T}\mathcal{A}\,\Phi$ is diagonal. Equation (61) then provides

$$\Phi^{\beta*T}\mathcal{B}\,\Phi^\alpha=-\lambda_\alpha\Phi^{\beta*T}\mathcal{A}\,\Phi^\alpha$$

so that

$$\Phi^{*T}\mathcal{B}\,\Phi=-(\Phi^{*T}\mathcal{A}\,\Phi)\Lambda \tag{64}$$

and Eq. (62) becomes

$$\dot{Y}=\Lambda Y+(\Phi^{*T}\mathcal{A}\,\Phi)^{-1}\mathcal{L} \tag{65}$$

Again the matrices in Eq. (65) must be truncated to be useful, and we must establish the commutativity of inversion and truncation for $\Phi^{*T}\mathcal{A}\,\Phi$. With the customary partitioning of Φ, however, this diagonal matrix becomes

$$\left[\begin{array}{c} \bar{\Phi}^{*T} \\ \hline \bar{\bar{\Phi}}^{*T} \end{array}\right][\mathcal{A}][\bar{\Phi}\ \bar{\bar{\Phi}}]=\left[\begin{array}{c|c} \bar{\Phi}^{*T}\mathcal{A}\bar{\Phi} & 0 \\ \hline 0 & \bar{\bar{\Phi}}^{*T}\mathcal{A}\bar{\bar{\Phi}} \end{array}\right]$$

so that the commutativity condition in Eq. (24) is trivially satisfied. Thus, we can, for practical computations, replace Eq. (65) with the truncated approximation

$$\dot{\bar{Y}}=\bar{\Lambda}\bar{Y}+(\bar{\Phi}^{*T}\mathcal{A}\,\bar{\Phi})^{-1}\bar{\Phi}^{*T}\mathcal{L} \tag{66}$$

The most *useful* conclusion of this study is the observation that the truncated Eq. (66) bears a relationship to the exact Eq. (65) that is *not* jeopardized by truncation/inversion commutativity constraints, although these constraints *do* jeopardize the utility of the alternatives represented by Eqs. (50) and (17). The apparent disadvantage of Eq. (66) lies in the fact that the quantities $\bar{Y}$, $\bar{\Lambda}$, and $\bar{\Phi}$ are complex. Equation (66) can, of course, be written in terms of its real and imaginary parts, and with the judicious use of orthogonality relationships the resulting equations can be greatly simplified. Because of the practical utility of the results, we construct in the following section the real equations implied by Eq. (66).

Real Modal Vibration Equations for Rotating Elastic Appendages

Because the matrix B in Eq. (3) is real, the eigenvalues $\lambda_1,\ldots\lambda_{12n}$ appear in complex conjugate pairs, and the corresponding pairs of eigenvectors are also complex conjugates. In the truncation operation, physical considerations require that all conjugate pairs be together either retained or rejected. It then becomes possible to group these pairs in such a way that, after truncation, Eq. (12) is replaced by

$$Q\cong[\Phi^1\Phi^2\ldots\Phi^N\Phi^{1*}\Phi^{2*}\ldots\Phi^{N*}]\begin{bmatrix} Y_1 \\ \vdots \\ Y_N \\ Y_1^* \\ \vdots \\ Y_N^* \end{bmatrix} \tag{67a}$$

$$\lambda_\alpha=i\sigma_\alpha \tag{67b}$$

and Eq. (66) becomes, with the imaginary eigenvalues

$$\begin{bmatrix} \dot{Y}_1 \\ \vdots \\ \dot{Y}_N \\ \dot{Y}_1^* \\ \vdots \\ \dot{Y}_N^* \end{bmatrix}=\begin{bmatrix} i\sigma_1 & & & & & 0 \\ & \ddots & & & & \\ & & i\sigma_N & & & \\ & & & -i\sigma_1 & & \\ & & & & \ddots & \\ 0 & & & & & -i\sigma_N \end{bmatrix}\begin{bmatrix} Y_1 \\ \vdots \\ Y_N \\ Y_1^* \\ \vdots \\ Y_N^* \end{bmatrix}$$

$$+\begin{bmatrix} (\phi^{1*T}\mathcal{A}\phi^1)^{-1} & & & & & 0 \\ & \ddots & & & & \\ & & (\phi^{N*T}\mathcal{A}\phi^N)^{-1} & & & \\ & & & (\phi^{1T}\mathcal{A}\phi^{1*})^{-1} & & \\ & & & & \ddots & \\ 0 & & & & & (\phi^{N^T}\mathcal{A}\phi^{N*})^{-1} \end{bmatrix}\begin{bmatrix} \phi^{1*T} \\ \vdots \\ \phi^{N*T} \\ \phi^{1T} \\ \vdots \\ \phi^{N^T} \end{bmatrix}\mathcal{L} \tag{68}$$

and since $\mathcal{A}^T = \mathcal{A}$ we have

$$\Phi^{\alpha^* T} \mathcal{A} \, \Phi^\alpha = \Phi^{\alpha T} \mathcal{A} \, \Phi^{\alpha^*}$$

Thus, the second set of N equations in Eq. (68) consists of equations conjugate to the first set, and we are left with the desired set of N first-order complex scalar equations typified by

$$\dot{Y}_\alpha = i\sigma_\alpha Y_\alpha + (\Phi^{\alpha^* T} \mathcal{A} \Phi^\alpha)^{-1} \Phi^{\alpha^* T} \mathcal{L} \quad \alpha = 1,\ldots,N \tag{69}$$

There remains the task of restating Eq. (69) as a pair of real equations, in terms of the variables in

$$Y_\alpha = \xi_\alpha + i\,\eta_\alpha \tag{70}$$

and the matrices Ψ^α and Γ^α defined by Eq. (37). By virtue of the orthogonality conditions in Ref. 10

$$\Phi^{\alpha^* T} \mathcal{A} \, \Phi^\alpha = 2\Psi^{\alpha^T} \, \mathcal{A} \, \Psi^\alpha \tag{71}$$

so that Eq. (69) becomes

$$\dot{\xi}_\alpha + i\dot{\eta}_\alpha = -\sigma_\alpha \eta_\alpha + i\sigma_\alpha \xi_\alpha$$
$$+ (2\Psi^{\alpha^T} \mathcal{A} \Psi^\alpha)^{-1} (\Psi^\alpha - i\Gamma^\alpha)^T \mathcal{L} \quad \alpha = 1,\ldots,N \tag{72}$$

which implies the real equations

$$\dot{\xi}_\alpha = -\sigma_\alpha \eta_\alpha + (2\Psi^{\alpha^T} \mathcal{A} \Psi^\alpha)^{-1} \Psi^{\alpha^T} \mathcal{L} \quad \alpha = 1,\ldots,N \tag{73a}$$

and

$$\dot{\eta}_\alpha = \sigma_\alpha \xi_\alpha - (2\Psi^{\alpha^T} \mathcal{A} \Psi^\alpha)^{-1} \Gamma^{\alpha^T} \mathcal{L} \quad \alpha = 1,\ldots,N \tag{73b}$$

Equations (73) can be simplified by normalizing the eigenvectors so that

$$\Psi^{\alpha^T} \, \mathcal{A} \, \Psi^\alpha = 1/2 \tag{73c}$$

Further simplification is afforded by the realization that, with Eq. (67b), Eq. (27) becomes

$$\phi^\alpha = \left[\begin{array}{c} \phi^\alpha \\ \hline i\sigma_\alpha \phi^\alpha \end{array}\right] \triangleq \left[\begin{array}{c} \psi^\alpha + i\gamma^\alpha \\ \hline i\sigma_\alpha (\psi^\alpha + i\gamma^\alpha) \end{array}\right]$$

which with Eq. (37) in turn implies

$$\Psi^\alpha = \left[\begin{array}{c} \psi \\ \hline -\sigma_\alpha \gamma^\alpha \end{array}\right] \qquad \Gamma^\alpha = \left[\begin{array}{c} \gamma^\alpha \\ \hline \sigma_\alpha \psi^\alpha \end{array}\right]$$

In view of these equations and the definition of $\mathcal{L}$ after Eq. (59), we can rewrite Eqs. (73) as the scalar equations

$$\dot{\xi}_\alpha = -\sigma_\alpha \eta_\alpha - \sigma_\alpha \gamma^{\alpha^T} F \quad \alpha = 1,\ldots,N \tag{74a}$$

and

$$\dot{\eta}_\alpha = \sigma_\alpha \xi_\alpha - \sigma_\alpha \psi^{\alpha^T} F \quad \alpha = 1,\ldots,N \tag{74b}$$

or as the $N \times 1$ matrix equations

$$\dot{\bar{\xi}} = -\bar{\sigma}\bar{\eta} - \bar{\sigma}\,\bar{\gamma}^T F \tag{74c}$$

and

$$\dot{\bar{\eta}} = \bar{\sigma}\bar{\xi} - \bar{\sigma}\,\bar{\psi}^T F \tag{74d}$$

It now becomes possible to return to Eqs. (2) and (67) and establish the relationship between the real quantities q, $\dot{q}$, ξ, and η as follows:

$$Q \triangleq \left[\begin{array}{c} q \\ \hline \dot{q} \end{array}\right] \cong \sum_{\alpha=1}^{N} (\Phi^\alpha Y_\alpha + \Phi^{\alpha^*} Y_\alpha^*)$$
$$= \sum_{\alpha=1}^{N} 2(\Psi^\alpha \xi_\alpha - \Gamma^\alpha \eta_\alpha) = 2 \sum_{\alpha=1}^{N}$$
$$\left[\begin{array}{c} \psi^\alpha \xi_\alpha - \gamma^\alpha \eta_\alpha \\ \hline -\sigma_\alpha \gamma^\alpha \xi_\alpha - \sigma_\alpha \psi^\alpha \eta_\alpha \end{array}\right]$$

or in equivalent matrix terms

$$q = 2(\bar{\psi}\bar{\xi} - \bar{\gamma}\bar{\eta}) \tag{75a}$$

and

$$\dot{q} = -2(\bar{\gamma}\bar{\sigma}\bar{\xi} + \bar{\psi}\bar{\sigma}\bar{\eta}) \tag{75b}$$

so that

$$\ddot{q} = -2(\bar{\gamma}\bar{\sigma}\dot{\bar{\xi}} + \bar{\psi}\bar{\sigma}\dot{\bar{\eta}}) \tag{75c}$$

In comparing these results with Eq. (45) and (46), we see that with the interpretation

$$Z = \left[\begin{array}{c} 2\xi \\ \hline -2\eta \end{array}\right]$$

we have in Eq. (75) obtained again the real transformation presented in Eq. (45), but instead of the truncated transformed equations with the troublesome inverse in Eq. (48) we have the much simpler results in Eq. (74).

Conclusions

We have established necessary and sufficient conditions for the commutativity of matrix inversion and truncation, and applied these criteria to both the most general problem of constantly rotating discretized flexible appendages and to the special case of the constantly rotating discretized elastic appendage with no energy dissipation. Specifically, we have identified in Eqs. (13, 14, 47, and 65) four transformed versions of the given Eq. (1) or its special case in Eq. (36), and in each case the truncated subset of the transformed equations has been tested for equivalence to an alternative version in which truncation precedes matrix inversion. This test is accomplished by testing the transformation matrix for commutativity of inversion and truncation.

In the most general case the commutativity condition presents a serious obstacle to computation, since the computationally attractive Eq. (17) is not equivalent to the formally truncated version of Eq. (14). We are obliged to adopt Eq. (16) as the truncated approximation of the exact Eq. (13), and this choice requires the computation of as many adjoint system eigenvectors as we choose to retain after truncation. Modern computational algorithms in the most general case do not yet permit convergence on preselected subsets of the eigenvalues (such as those of lowest frequency) although this is possible in an increasing range of special cases. For the most general case with present algorithms, we would be obliged to calculate all eigenvalues but only the selected $2N$ eigenvectors of the adjoint system.

For the special case of the undamped, rotating, elastic appendage one can use a certain transformation from the literature[10] with impunity, while another published transformation is jeopardized in its validity by its failure to meet the commutativity condition. More specifically, for the special case of the rotating elastic appendage, commutativity conditions fail to justify the adoption of the computationally attractive Eq. (50) as a truncated approximation of the real Eq. (47), and to meet the commutativity test we must accept instead the complex transformation leading to Eq. (66) as an approximation of the exact Eq. (65). Because of the practical utility of Eq. (66), it is expanded here in real terms, culminating in Eq. (74).

The commutativity conditions developed here as Eqs. (19) and (24) are, of course, applicable to other matrices than those considered here, and they may prove to be useful in the evaluation of other transformations than those examined in this paper.

Appendix A. Proof of Commutativity Theorem

Assume initially that a and d in Eq. (18) are nonsingular. Then[11]

$$m^{-1}=\left[\begin{array}{c|c}(a-bd^{-1}c)^{-1} & -a^{-1}b(d-ca^{-1}b)^{-1}\\ \hline -d^{-1}c(a-bd^{-1}c)^{-1} & (d-ca^{-1}b)^{-1}\end{array}\right]$$

and

$$\overline{(m^{-1})}=[(a-bd^{-1}c)^{-1} \;|\; -a^{-1}b(d-ca^{-1}b)^{-1}]$$

while

$$\bar{m}\dagger \triangleq \left([a^T \;|\; c^T]\left[\begin{array}{c}a\\ \hline c\end{array}\right]\right)^{-1}[a^T \;|\; c^T]$$

$$=[a^Ta+c^Tc]^{-1}[a^T \;|\; c^T]$$

$$=[(a^Ta+c^Tc)^{-1}a^T \;|\; (a^Ta+c^Tc)^{-1}c^T]$$

As necessary and sufficient conditions for Eq. (20) we have

$$(a^Ta+c^Tc)^{-1}a^T=(a-bd^{-1}c)^{-1} \tag{A1}$$

and

$$(a^Ta+c^Tc)^{-1}c^T=-a^{-1}b(d-ca^{-1}b)^{-1} \tag{A2}$$

Premultiplying Eq. (A1) by (a^Ta+c^Tc) and postmultiplying Eq. (A1) by $(a-bd^{-1}c)$ yields

$$a^T(a-bd^{-1}c)=aa^T+c^Tc$$

or

$$a^Tbd^{-1}c+c^Tc=0$$

or

$$(a^Tb+c^Td)d^{-1}c=0 \tag{A3}$$

Similarly, from Eq. (A2)

$$c^T(d-ca^{-1}b)=-(a^Ta+c^Tc)a^{-1}b$$

or

$$a^Tb+c^Td=0 \tag{A4}$$

Satisfaction of Eq. (19) is sufficient for Eq. (A3) and both necessary and sufficient for Eq. (A4), so that the theorem is proved for the special case of nonsingular partitions a and d.

If a and/or d in the given partitioning of m is singular, the theorem can be proved in a manner to be developed here explicitly only for singular a and nonsingular d.

Let a be a singular $r\times r$ matrix, and let m be of dimension $s\times s$. Since m is nonsingular, the truncated matrix $\bar{m}$ has r independent columns, and at least r independent rows. Consequently, one can always interchange rows of $\bar{m}$ to obtain a nonsingular upper $r\times r$ row partition, to be called a_0. The interchanging operation can be accomplished formally by premultiplying by a symmetric matrix V typified for the interchange of a single pair of rows by[12]

$$V=\left[\begin{array}{c|c|c|c|c}U_i & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 1 & 0\\ \hline 0 & 0 & U_j & 0 & 0\\ \hline 0 & 1 & 0 & 0 & \\ \hline 0 & 0 & 0 & & U_k\end{array}\right] \tag{A5}$$

in which U_p is the unit matrix of dimension $p\times p$ for any p, and the isolated scalar 1 appears in the row numbers of m which are interchanged by V_m. If several pairs of rows are to be interchanged in m, the corresponding rows of V are modified according to the pattern just shown. Note that V is a kind of *elementary* matrix, equal to its own inverse, so that $VV=U_r$. Now by definition

$$Vm=\left[\begin{array}{c|c}a_0 & b_0\\ \hline c_0 & d_0\end{array}\right]$$

and both a_0 and d_0 are nonsingular. Thus, the preceding proof established that

$$a_0^Tb_0+c_0^Td_0=0 \tag{A6}$$

is necessary and sufficient for

$$[\overline{(Vm)}]\dagger=[\overline{(Vm)^{-1}}] \tag{A7}$$

However,

$$[\overline{(Vm)}]\dagger=[V\bar{m}]\dagger=[(V\bar{m})^T(V\bar{m})]^{-1}(V\bar{m})^T$$

$$=[\bar{m}^TV^TV\bar{m}]^{-1}\bar{m}^TV=(\bar{m}^T\bar{m})^{-1}\bar{m}^TV=\bar{m}\dagger V$$

and

$$[\overline{(Vm)^{-1}}]=[\overline{m^{-1}V}]=\overline{(m^{-1})}V$$

so that Eq. (A7) is equivalent to

$$\bar{m}\dagger V=\overline{(m^{-1})}V \tag{A8}$$

Since postmultiplication by V merely interchanges columns, this result leads to Eq. (20).

Moreover, with the proper partitioning of V we find

$$Vm=\left[\begin{array}{c|c}V_{11} & V_{12}\\ \hline V_{21} & V_{22}\end{array}\right]\left[\begin{array}{c|c}a & b\\ \hline c & d\end{array}\right]=\left[\begin{array}{c|c}a_0 & b_0\\ \hline c_0 & d_0\end{array}\right]$$

Hence

$$a_0^Tb_0+c_0^Td_0=(V_{11}a+V_{12}c)^T(V_{11}b+V_{12}d)$$

$$+(V_{21}a+V_{22}c)^T(V_{21}b+V_{22}d)=a^TV_{11}^TV_{11}b$$

$$+c^TV_{12}^TV_{11}b+a^TV_{11}^TV_{12}d+c^TV_{12}^TV_{12}d+a^TV_{21}^TV_{21}b$$

$$+c^TV_{22}V_{21}b+a^TV_{21}^TV_{22}d+c^TV_{22}^TV_{22}d$$

With the identities

$$V_{11}^T\,V_{12}=0 \qquad V_{22}^TV_{21}=0$$

$$V_{11}^TV_{11}+V_{21}^TV_{21}=U_p \qquad V_{22}^TV_{22}+V_{12}^TV_{12}=U_{s-r}$$

we then have

$$a_0^T b_0 + c_0^T d_0 = a^T b + c^T d \tag{A9}$$

Thus, the theorem is proved when a is singular by the combination of Eqs. (A6-A9). Similar operations accommodate the case when d is singular, or when d and a are both singular, completing the general proof of the stated theorem.

Appendix B. Proof of Commutativity Corollary

Define a as the uppermost $r \times r$ partition of a_1. Comparison of Eqs. (18) and (23) then permits the identification of some submatrices a_2 and b_2 such that

$$a_1 = \begin{bmatrix} a \\ \hline a_2 \end{bmatrix} \qquad b_1 = \begin{bmatrix} b \\ \hline b_2 \end{bmatrix}$$

$$c = \begin{bmatrix} a_2 \\ \hline c_1 \end{bmatrix} \qquad d = \begin{bmatrix} b_2 \\ \hline d_1 \end{bmatrix}$$

Now

$$a_1^T b_1 + c_1^T d_1 = \begin{bmatrix} a \\ \hline a_2 \end{bmatrix}^T \begin{bmatrix} b \\ \hline b_2 \end{bmatrix} + c_1^T d_1 = a^T b + a_2^T b_2 + c_1^T d_1$$

$$= a^T b + \begin{bmatrix} a_2 \\ \hline c_1 \end{bmatrix}^T \begin{bmatrix} b_2 \\ \hline d_1 \end{bmatrix} = a^T b + c^T d \tag{B1}$$

Hence, by Eq. (B1), Eq. (24) implies Eq. (19), which is necessary and sufficient for the validity of Eqs. (20) and (25), proving the corollary.

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